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EXAMPLES OF $\mathscr{L}_p$ SPACES $(1 < p \neq 2 < \infty)^{\dagger}$

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ABSTRACT
We present a simple method for constructing new $\mathscr{L}_p$ spaces $(1 < p \neq 2 < \infty)$ out of old ones. Using this method and results of H. P. Rosenthal we prove the existence of a sequence of mutually nonisomorphic separable infinite dimensional $\mathscr{L}_p$ spaces $(1 < p \neq 2 < \infty)$.

1. Introduction

Since the theory of $\mathscr{L}_p$ spaces was introduced in [3], [5] it was an open problem whether there exist infinitely many mutually nonisomorphic separable infinite dimensional $\mathscr{L}_p$ spaces $(1 < p \neq 2 < \infty)$ (The case $p = 1$ was solved in [2]). The purpose of this paper is to solve this problem.

THEOREM. *There exists a sequence of mutually nonisomorphic, separable, infinite dimensional $\mathscr{L}_p$ spaces* $(1 < p \neq 2 < \infty)$.

The only knowledge required from the theory of $\mathscr{L}_p$ spaces is the fact proved in [3], [5] that X is a separable $\mathscr{L}_p$ space $(1 < p \neq 2 < \infty)$ if and only if it is isomorphic to a complemented subspace of $L_p(I)$ $(I = (0,1))$ and it is not isomorphic to l_2.

The proof of the theorem is carried out by introducing a very simple method to construct new $\mathscr{L}_p$ spaces out of old ones and combining this method with results of H. P. Rosenthal [9] concerning the span in L_p of a sequence of independent random variables.

The notations are standard and those which are not explained here can be found in [6].

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We only mention that $(r_i)_{i=1}^{\infty}$ is the sequence of the Rademacher functions.

I denotes the unit interval.

The symbol $\approx$ ($\overset{K}{\approx}$) denotes inequalities in both sides with constants which do not depend on the scalars (respectively – with constants K and K^{-1}).

2. Preliminaries

DEFINITION. Let $(X_i)_{i=1}^{n}$ be a finite sequence of subspaces of $L_p(I)$ $(1 \leqq p < \infty)$; we define $X_1 \otimes X_2 \otimes \cdots \otimes X_n = \otimes_{i=1}^{n} X_i$ to be the closed linear span in $L_p(I^n)$ of functions of the form $(x_1 \otimes \cdots \otimes x_n)(t_1, t_2, \cdots, t_n) = x_1(t_1) \cdot x_2(t_2) \cdots x_n(t_n)$; $x_i \in X_i$. (This definition coincides with the completion of the usual tensor product in a certain norm.)

LEMMA 1. *Let X_i $i = 1, 2, \cdots, n$ be complemented subspaces of $L_p(I)$ $(1 \leqq p < \infty)$; then $\otimes_{i=1}^{n} X_i$ is complemented in $L_p(I^n)$.*

LEMMA 2. *Let X_i, Y_i, $i = 1, 2, \cdots, n$, be subspaces of $L_p(I)$ and let $T_i : X_i \to Y_i$ be isomorphisms onto, $i = 1, 2, \cdots, n$. Then $\otimes_{i=1}^{n} T_i : \otimes_{i=1}^{n} X_i \xrightarrow{\text{onto}} \otimes_{i=1}^{n} Y_i$ is an isomorphism.*

LEMMA 3. *Let X_i, $i = 1, 2, \cdots, n$, be subspaces of $L_p(I)$ $(1 \leqq p < \infty)$ with unconditional bases $(x_j^i)_{j=1}^{\infty}$, $i = 1, 2, \cdots, n$ respectively. Then $(x_{j_1}^1 \otimes x_{j_2}^2 \otimes \cdots x_{j_n}^n)_{j_1, \cdots, j_n = 1}^{\infty}$ constitutes an unconditional basis for $\otimes_{i=1}^{n} X_i$.*

Lemmas 1 and 2 are well known and easy to prove. Let us just mention that if, for example, $n = 2$ and $P_i : L_p(I) \xrightarrow{\text{onto}} X_i$, $i = 1, 2$, are the given projections in Lemma 1 and if $k(u, v)$ is a continuous function then the projection of k into $X_1 \otimes X_2$ is given in the following way. Fix v and let $h(\cdot, v)$ be $P_1(k(\cdot, v))$. Now consider $h(u, v)$ as a representing function of its equivalent class; then for almost every $u \in I$, $h(u, \cdot) \in L_p(I)$. Apply P_2 to this function.

PROOF OF LEMMA 3. The proof is carried out by induction. We shall consider only the case $n = 2$. The induction step is carried out in a similar manner.

It is obvious that $\overline{\text{span}}\,[(x_i^1 \otimes x_j^2)_{i,j=1}^{\infty}] = X_1 \otimes X_2$. Now by the unconditionality of $(x_i^1)_{i=1}^{\infty}$, $(x_i^2)_{i=1}^{\infty}$, by the generalization of Khinchine's inequality for expressions of the form $\int_0^1 \int_0^1 | \sum_{i,j=1}^{\infty} b_{i,j} r_i(t) r_j(s) |^p dt ds$ and by Fubini's theorem we get that for all scalars $(a_{i,j})_{i,j=1}^{\infty}$,

$$\left\| \sum_{i,j=1}^{\infty} a_{i,j} x_i^1 \otimes x_j^2 \right\|^p = \int_0^1 \int_0^1 \left| \sum_{i,j=1}^{\infty} a_{i,j} x_i^1(t) x_j^2(s) \right|^p dtds$$

$$\approx \int_0^1 \int_0^1 \int_0^1 \left| \sum_{i,j=1}^{\infty} a_{i,j} x_i^1(t) x_j^2(s) r_i(u) \right|^p dudtds$$

$$\approx \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left| \sum_{i,j=1}^{\infty} a_{i,j} x_i^1(t) x_j^2(s) r_i(u) r_j(v) \right|^p dudvdtds$$

$$\approx \int_0^1 \int_0^1 \left(\sum_{i,j=1}^{\infty} \left| a_{i,j} x_i^1(t) x_j^2(s) \right|^2 \right)^{p/2} dtds.$$

Thus, $(x_i^1 \otimes x_j^2)_{i,j=1}$ constitutes an unconditional basis. ∎

Proposition 1. *Let $1 \leqq p < r_1 < r_2 < \cdots < r_n \leqq 2$ and let X_i, $i = 1, 2, \cdots, n$, be subspaces of $L_p(I)$ such that X_i is isomorphic to l_{r_i}, $i = 1, 2, \cdots, n$. Then the natural basis of $\bigotimes_{i=1}^n X_i$ is equivalent to the natural basis of the space*:

$$X(r_1, r_2, \cdots, r_n) = \{(a_{i_1,\cdots,i_n})_{i_1,\cdots,i_n=1}^{\infty}; \|(a_{i_1,\cdots,i_n})\|$$

$$= \left(\sum_{i_1=1}^{\infty} \left(\sum_{i_2=1}^{\infty} \cdots \left(\sum_{i_{n-1}=1}^{\infty} \left(\sum_{i_n=1}^{\infty} |a_{i_1,\cdots,i_n}|^{r_n} \right)^{r_{n-1}/r_n} \right)^{r_{n-2}/r_{n-1}} \cdots \right)^{r_1/r_2} \right)^{1/r_1} \Bigg\}.$$

Proof. By Lemma 2 one can assume without loss of generality that for all $i = 1, \cdots, n$, X_i is spanned by independent random variables $(x_j^i)_{j=1}^{\infty}$, each being r_i stable. Thus, for all $t < r_i$, $(x_j^i)_{j=1}^{\infty}$ is equivalent to the usual basis of l_{r_i}. (For the definition and properties of r-stable random variables see e.g. [9].)

We shall prove the proposition by induction. For $n = 1$ there is nothing to prove. Assume that the proposition is true for $n = k - 1$ and all $1 \leqq p \leqq 2$, then:

$$\left\| \sum_{i_1,\cdots,i_k=1}^{\infty} a_{i_1,\cdots,i_k} x_{i_1}^1 \otimes \cdots \otimes x_{i_k}^k \right\|^p$$

$$= \int_0^1 \cdots \int_0^1 \left| \sum_{i_1,\cdots,i_k=1}^{\infty} a_{i_1,\cdots,i_k} x_{i_1}^1(t_1) \cdots x_{i_k}^k(t_k) \right|^p dt_1 \cdots dt_k$$

$$\approx \int_0^1 \cdots \int_0^1 \left(\sum_{i_1=1}^{\infty} \left| \sum_{i_2,\cdots,i_k=1}^{\infty} a_{i_1,\cdots,i_k} x_{i_2}^2(t_2) \cdots x_{i_k}^k(t_k) \right|^{r_1} \right)^{p/r_1} dt_2 \cdots dt_k$$

$$\leqq \left(\int_0^1 \cdots \int_0^1 \sum_{i_1=1}^{\infty} \left| \sum_{i_2,\cdots,i_k=1}^{\infty} a_{i_1,\cdots,i_k} x_{i_2}^2(t_2) \cdots x_{i_k}^k(t_k) \right|^{r_1} dt_2 \cdots dt_k \right)^{p/r_1}$$

$$\approx \|(a_{i_1,\cdots,i_k})\|_{X(r_1,\cdots,r_k)}^p.$$

(We used the fact that $f(s) = s^{p/r_1}$ is concave and the induction hypothesis for r_1 instead of p.)

On the other hand we get by the induction hypothesis and the generalization of the triangle inequality to integrals (instead of sums) that:

$$\left\| \sum_{i_1,\dots,i_k=1}^{\infty} a_{i_1,\dots,i_k} x^1_{i_1} \otimes \cdots \otimes x^k_{i_k} \right\|^p$$

$$= \int_0^1 \cdots \int_0^1 \left| \sum_{i_1,\dots,i_k=1}^{\infty} a_{i_1,\dots,i_k} x^1_{i_1}(t_1) \cdots x^k_{i_k}(t_k) \right|^p dt_1 \cdots dt_k$$

$$\approx \int_0^1 \left\| \sum_{i_k=1}^{\infty} (a_{i_1,\dots,i_k} x^k_{i_k}(t_k)) \right\|^p_{X(r_1,\dots,r_{k-1})} dt_k$$

$$= \int_0^1 \left\| \left(\left| \sum_{i_k=1}^{\infty} a_{i_1,\dots,i_k} x^k_{i_k}(t_k) \right|^p \right) \right\|_{X(r_1/p,\, r_2/p,\dots,r_{k-1}/p)} dt_k$$

$$\geqq \left\| \left(\int_0^1 \left| \sum_{i_k=1}^{\infty} a_{i_1,\dots,i_k} x^k_{i_k}(t_k) \right|^p dt_k \right) \right\|_{X(r_1/p,\dots,r_{k-1}/p)}$$

$$= \left\| \left(\left(\sum_{i_k=1}^{\infty} \left| a_{i_1,\dots,i_k} \right|^{r_n} \right)^{p/r_n} \right) \right\|_{X(r_1/p,\dots,r_{k-1}/p)}$$

$$= \left\| (a_{i_1,\dots,i_k}) \right\|^p_{X(r_1,\dots,r_k)}.$$

■

3. Proof of the theorem

We shall need the following:

PROPOSITION 2. *Let $p < r < s \leqq 2$. Then there does not exist a sequence $(x_{i,j})_{i,j=1}^{\infty}$ of independent random variables on $(0,1)$ such that the $(x_{i,j})_{i,j=1}^{\infty}$ considered as elements in $L_p(I)$ are equivalent to the usual basis of $(l_s \oplus l_s \oplus \cdots)_r = X(r,s)$.*

We shall assume the validity of Proposition 2 for the moment and pass to the

PROOF OF THE THEOREM. It is clearly sufficient to prove the theorem for the case $1 < p < 2$.

Let X_p be Rosenthal's space ([8], [9]). Then there exists a sequence $(x_i)_{i=1}^{\infty}$ of independent symmetric 3-valued random variables in $L_p(I)$ that constitutes a basis for X_p. It is known ([8], [9], [10]) that X_p is complemented in L_p and it contains an isomorph of l_r for each $p < r \leqq 2$; thus $X_p^{\otimes n} = X_p \otimes \cdots \otimes X_p$ (n times) contains an isomorph of $l_{r_1} \otimes l_{r_2} \otimes \cdots \otimes l_{r_n}$ for all $p < r_1 < r_2 < \cdots < r_n \leqq 2$.

We shall prove by induction that $X_p^{\otimes n}$ does not contain an isomorph of $l_{r_1} \otimes \cdots \otimes l_{r_{2n}}$ for any $p < r_1 < \cdots < r_{2n} \leqq 2$. This will imply that $X_p^{\otimes 2n}$ is not isomorphic to a subspace of $X_p^{\otimes k}$, $k \leqq n$, and thus X_p, $X_p^{\otimes 2}$, $X_p^{\otimes 4}$, $X_p^{\otimes 8}, \cdots$ constitutes a sequence of infinite dimensional, separable, mutually nonisomorphic $\mathscr{L}_p$ spaces.

The case $n = 1$ follows immediately from Proposition 2 and the simple fact that the usual basis of $(l_s \oplus l_s \oplus \cdots)_r$ is reproducible in the sense of [4].

Assume that the assertion above is true for $n = k - 1$ and assume that $l_{r_1} \otimes \cdots \otimes l_{r_{2k}}$ is isomorphic to a subspace of $X_p^{\otimes k}$ where $p < r_1 < \cdots < r_{2k} \leqq 2$. Let $(y_{i_1, \cdots, i_{2k}})_{i_1, \cdots, i_{2k} = 1}^{\infty}$ denote the image under this isomorphism of the usual basis of $l_{r_1} \otimes \cdots \otimes l_{r_{2k}}$.

Let

$$\varphi : N \to N \times N \text{ be onto and one to one.}$$

Let

$$P_m : X_p^{\otimes k} \xrightarrow{\text{onto}} [x_{i_1} \otimes \cdots \otimes x_{i_k}]_{i_1, \cdots, i_k = 1}^{m}$$

and

$$Q_m : X_p^{\otimes k} \xrightarrow{\text{onto}} [x_{i_1} \otimes \cdots \otimes x_{i_k}]_{i_1, \cdots, i_k = m+1}^{\infty}$$

be the natural projections.

It is known [8] that X_p is isomorphic to $X_p \oplus X_p$. Thus, $X_p^{\otimes (k-1)}$ is isomorphic to $\Sigma_{i=1}^{2^{(k-1)l}} \oplus X_p^{\otimes (k-1)}$ for each l. Hence, for each m, $(I - Q_m) X_p^{\otimes k}$, being isomorphic to a finite direct sum of copies of $X_p^{\otimes (k-1)}$, is isomorphic to a subspace of $X_p^{\otimes (k-1)}$.

Let $(\varepsilon_i)_{i=1}^{\infty}$ be a decreasing sequence tending to zero. Choose $z_1 \in [y_{\varphi(1), i_3, \cdots, i_{2k}}]_{i_3, \cdots, i_{2k} = 1}^{\infty}$, $\| z_1 \| = 1$. There exists an m_1 such that $\| (I - P_{m_1}) z_1 \| < \varepsilon_1$. $Y_2 = [y_{\varphi(2), i_3, \cdots, i_{2k}}]_{i_3, \cdots, i_{2k} = 1}^{\infty}$ is isomorphic to $l_{r_3} \otimes \cdots \otimes l_{r_{2k}}$ and thus, by the induction hypothesis and the fact mentioned above that $(I - Q_{m_1}) X_p^{\otimes k}$ is isomorphic to a subspace of $X_p^{\otimes (k-1)}$, it follows that $(I - Q_{m_1})_{|Y_2}$ is not an isomorphism. Thus there exists a $z_2 \in Y_2$ such that $\| z_2 \| = 1$ and $\| (I - Q_{m_1}) z_2 \| < \varepsilon_2 / 2$. Choose now m_2, $m_2 > m_1$, such that $\| (I - P_{m_2}) z_2 \| < \varepsilon_2 / 2$.

Continuing this way we get a sequence $(z_l)_{l=1}^{\infty}$ such that $\| z_l \| = 1$ and $z_l \in [y_{\varphi(l), i_3, \cdots, i_{2k}}]_{i_3, \cdots, i_{2k} = 1}^{\infty}$, $l = 1, 2, \cdots$, and an increasing sequence $(m_l)_{l=1}^{\infty}$ of natural numbers such that $(Q_{m_0} = 0)$

$$\| (I - Q_{m_{l-1}}) z_l \| < \varepsilon_l / 2 ; \| (I - P_{m_l}) z_l \| < \varepsilon_l / 2, \; l = 1, 2, \cdots .$$

Clearly if the ε_l are small enough then the sequence $(z_l)_{l=1}^{\infty}$ is equivalent to $(Q_{m_{l-1}} P_{m_l} z_l)_{l=1}^{\infty}$. On the other hand it is easy to see using Proposition 1 that, since

$z_l \in [y_{\varphi(l),i_3,\cdots,i_{2k}}]_{i_3,\cdots,i_{2k}=1}^{\infty}$, the sequence $(z_l)_{l=1}^{\infty}$ is equivalent to a permutation of the usual basis of $l_{r_1} \otimes l_{r_2}$. Observe also that $(Q_{m_l}\,{}_1 P_{m_l} z_l)_{l=1}^{\infty}$ is a sequence of independent random variables and this contradicts Proposition 2. This contradiction concludes the proof of the induction step and thus the proof of the theorem. ∎

REMARK 1. It is easy to show that the fact that in the sequence $(X_p^{\otimes 2^k})_{k=0}^{\infty}$ each member is not isomorphic to a subspace of its previous ones implies that the same is true for the whole sequence $(X_p^{\otimes n})_{n=1}^{\infty}$. Indeed assume that $n < m$ and $X_p^{\otimes m}$ is isomorphic to a subspace of $X_p^{\otimes}$, then for all $k \geqq 0$, $X_p^{\otimes [m+(k+1)(m-n)]}$ is isomorphic to a subspace of $X_p^{\otimes [n+(k+1)(m-n)]} = X_p^{\otimes [m+k(m-n)]}$. By induction, it follows that for every $k \geqq 0$, $X_p^{\otimes [m+k(m-n)]}$ is isomorphic to a subspace of $X_p^{\otimes n}$. Now choose natural numbers s, t, k such that $n \leqq 2^s < 2^t \leqq m + k(m-n)$. Then we get that $X_p^{\otimes 2^t}$ is isomorphic to a subspace of $X_p^{\otimes 2^s}$.

DEFINITION. Let $1 \leqq p < \infty$, $\varepsilon > 0$. A sequence $(x_i)_{i=1}^{\infty}$ of independent random variables in $L_p(I)$ is said to be (p, ε)-equi-distributed if there exists a sequence $(z_i)_{i=1}^{\infty}$ of equi-distributed simple independent random variables such that $\|x_i - z_i\| < \varepsilon$.

In the proof of Proposition 2 we shall use the following simple and essentially well known

LEMMA 4. *Let $1 \leqq p < \infty$, $\varepsilon > 0$ and let $(x_i)_{i=1}^{\infty}$ be a sequence of p-equi-integrable independent random variables in $L_p(I)$. Then there exists a subsequence $(x_{i_n})_{n=1}^{\infty}$ of $(x_i)_{i=1}^{\infty}$ which is (p, ε)-equi-distributed.*

Recall that a sequence $(x_i)_{i=1}^{\infty} \subset L_p(I)$ is p-equi-integrable if for every $\varepsilon > 0$ there exists an N so that $\int_{A_i} |x_i(t)|^p dt < \varepsilon$, for all i, where $A_i = \{t\,; |x_i(t)| \geqq N\}$.

PROOF OF LEMMA 4. Let N be such that

$$\int_{\{|x_n(t)| \geqq N\}} |x_n(t)|^p \, dt < \varepsilon^p \quad n = 1, 2, \cdots .$$

Divide the interval $[-N, N)$ into a finite sequence of disjoint intervals $(I_i)_{i=1}^{k} = ([a_i, b_i))_{i=1}^{k}$ of diameter at most ε. Let $\delta > 0$ be such that $\lambda(A) < \delta$ implies $\int_A |x_n(t)|^p dt < \varepsilon^p / k$, where λ denotes the Lebesgue measure on the unit interval. There exists a subsequence $(x_{n_l})_{l=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ and numbers α_i, $i = 1, 2, \cdots, k$, such that

$$\lambda(x_{n_l} \in I_i) \xrightarrow[l \to \infty]{} \alpha_i$$

and
$$|\lambda(x_{n_l} \in I_i) - \alpha_i| < \delta/2.$$

Let $\beta_i = \alpha_i - \delta/2$, $i = 1, 2, \cdots, k$. Then for each l,

$$\beta_i + \delta > \lambda(x_{n_l} \in I_i) > \beta_i.$$

Define random variables z_{n_l}, $l = 1, 2, \cdots$, in the following manner: z_{n_l} equals a_i on a set A_i^l contained in $(x_{n_l} \in I_i)$ of measure β_i, $i = 1, 2, \cdots, k$, and z_{n_l} equals zero on $I \setminus \bigcup_{i=1}^{k} A_i^l$. One can easily show that the z_{n_l} can be chosen to be independent. (One can assume without loss of generality that the x_n are defined on $I^{\aleph_0}$ and x_n depends only on the nth coordinate. The z_{n_l} can be chosen then to depend on the n_lth coordinate only.)

Now, all the z_{n_l} have the same distribution and

$$\|z_{n_l} - x_{n_l}\|_l^p \leqq \int_{\{|x_{n^l}| \geqq N\}} |x_{n_l}|^p + \sum_{i=1}^{k} \int_{A_i^l} |z_{n_l} - x_{n_l}|^p + \sum_{i=1}^{k} \int_{\{x_{n^l} \in I_i\} \setminus A_i^l} |x_{n_l}|^p$$
$$\leqq \varepsilon^p + \varepsilon^p + \sum_{i=1}^{k} \varepsilon^p / k = 3 \cdot \varepsilon^p.$$

Thus $(x_{n_l})_{l=1}^{\infty}$ is $(p, 3^{1/p} \cdot \varepsilon)$-equi-distributed. ∎

PROOF OF PROPOSITION 2. Assume that $(x_{i,j})_{i,j=1}^{\infty}$ is a sequence of independent random variables which is equivalent to the usual basis of $(l_s \oplus l_s \oplus \cdots)_r$, i.e. there exists a K such that

$$\left\| \sum_{i,j=1}^{\infty} a_{i,j} x_{i,j} \right\| \overset{K}{\approx} \left(\sum_i \left(\sum_j |a_{i,j}|^s \right)^{r/s} \right)^{1/r}.$$

$[x_{i,j}]_{i,j=1}^{\infty}$ does not contain an isomorph of l_p and thus $(x_{i,j})_{i,j=1}^{\infty}$ is p-equi-integrable (cf., e.g. [1]). By passing to a subsequence of $(x_{i,j})_{j=1}^{\infty}$ for each i one can assume without loss of generality that $(x_{i,j})_{j=1}^{\infty}$ is $(p, 2^{-i})$-equi-distributed. Moreover, by the reasoning used in the proof of Lemma 4, we may assume also that there exist simple independent random variables $(z_{i,j})_{i,j=1}^{\infty}$ such that $\|z_{i,j} - x_{i,j}\| < 2^{-i}$, $i, j = 1, 2, \cdots$.

By using Lemma 4 repeatedly and then the diagonal procedure it follows that there exists a subsequence $(x_{i_n,1})_{n=1}^{\infty}$ of $(x_{i,1})_{i=1}^{\infty}$ such that $(x_{i_n,1})_{n=m}^{\infty}$ is $(p, 2^{-m})$-equi-distributed, i.e. for each $m = 1, 2, \cdots$ there exists a sequence of simple equi-distributed independent random variables $(w_{n,1}^m)_{n=m}^{\infty}$ such that $\|w_{n,1}^m - x_{i_n,1}\| < 2^{-m}$, $n = m, m+1, \cdots$. We assume also that the $(w_{n,1}^m)_{n=m}^{\infty}$ are independent of $(x_{i,j}, z_{i,j})_{i=1,j=2}^{\infty, \infty}$, and that $i_n = n$, $n = 1, 2, \cdots$.

It is well known and simple to prove (see for example [9]) that if x, x_1, y are simple random variables such that $\operatorname{dist} x = \operatorname{dist} y$ then there exists a random variable y_1 such that $\operatorname{dist}(x, x_1) = \operatorname{dist}(y, y_1)$. Thus, for each m, $m \leqq i$ and $2 \leqq j$ we can find a random variable $w_{i,j}^m$ such that $\operatorname{dist}(z_{i,1} w_{i,1}^m) = \operatorname{dist}(z_{i,j}, w_{i,j}^m)$. The $w_{i,j}^m$ can clearly be chosen so that for every m the variables $(w_{i,j}^m)_{i=m,j=1}^{\infty}$ are independent.

Now, for all j, $m = 1, 2, \cdots$ and $i = m, m+1, \cdots$,

$$\begin{aligned}\| w_{i,j}^m - x_{i,j} \| &\leqq \| w_{i,j}^m - z_{i,j} \| + \| z_{i,j} - x_{i,j} \| \\ &= \| w_{i,1}^m - z_{i,1} \| + \| z_{i,j} - x_{i,j} \| \\ &\leqq \| w_{i,1}^m - x_{i,1} \| + \| x_{i,1} - z_{i,1} \| + \| z_{i,j} - x_{i,j} \| \leqq 3 \cdot 2^{-m} .\end{aligned}$$

Fix an integer l, a permutation π of $(1, 2, \cdots, l) \times (1, 2, \cdots, l)$ and scalars $(a_{i,j})_{i,j=1}^{l}$. We shall show that

$$\left\| \sum_{i,j=1}^{l} a_{i,j} x_{i,j} \right\| \overset{K^4}{\approx} \left\| \sum_{i,j=1}^{l} a_{\pi(i,j)} x_{i,j} \right\| .$$

This will imply that $(x_{i,j})_{i,j=1}^{\infty}$ is a symmetric basic sequence which is a contradiction to the fact that it is equivalent to the usual basis of $(l_s \oplus l_s \oplus \cdots)_r$.

For each $m = 1, 2, \cdots$,

$$\left\| \sum_{i,j=1}^{l} a_{i,j} x_{i,j} \right\| \overset{K^2}{\approx} \left\| \sum_{i,j=1}^{l} a_{i,j} x_{m+i, m+j} \right\| .$$

The difference between the last expression and $\| \sum_{i,j=1}^{l} a_{i,j} w_{m+i,m+j}^m \|$ is less than or equal to $3 \cdot 2^{-m} \sum_{i,j=1}^{l} | a_{i,j} |$ which by enlarging m can be made as small as we wish.

The same considerations hold for $\| \sum_{i,j=1}^{l} a_{\pi(i,j)} x_{i,j} \|$ and

$$\left\| \sum_{i,j=1}^{l} a_{i,j} w_{[(m,m)+\pi^{-1}(i,j)]}^m \right\| = \left\| \sum_{i,j=1}^{l} a_{i,j} w_{m+i,m+j}^m \right\| .$$

(The last equality follows from the equi-distribution of the $w_{i,j}^m$.) This concludes the proof of the proposition. ∎

Remark 2. Professor H. P. Rosenthal informed the author that he has also proved this proposition (unpublished). The proof of Rosenthal involves the following two steps: (1) For $r < s$ $(l_s \oplus l_s \oplus \cdots)_r$ is not a modular sequence; (2) The span of a sequence of independent random variables in L_p is isomorphic to a modular sequence space (cf. [10] for the notion of modular sequence space).

4. Remarks and open problems

We first wish to give concrete representation to some of the spaces which can be constructed by the method of Section 2.

(a) If X, Y are subspaces of $L_p(I)$ $(1 \leqq p < \infty)$ and Y is isomorphic to l_p then $X \otimes Y$ is isomorphic to $(X \oplus X \oplus \cdots)_p$.

(b) If X, Y are subspaces of $L_p(I)$ and X is isomorphic to l_2 then $X \otimes Y$ is isomorphic to $\mathrm{Rad}(Y)$ (cf. [7]). If in addition Y has an unconditional basis $(y_i)_{i=1}^{\infty}$ then $X \otimes Y$ is isomorphic to $(l_2 \oplus l_2 \oplus \cdots)_{(y_i)}$ i.e. to the space of all sequences $(a_{i,j})_{i,j=1}^{\infty}$ with

$$\|(a_{i,j})_{i,j=1}^{\infty}\| = \left|\left|\left| \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{i,j}|^2 \right)^{1/2} y_i \right|\right|\right| < \infty$$

where $|||\cdot|||$ is an equivalent norm on Y in which $(y_i)_{i=1}^{\infty}$ has unconditional constant one.

(c) Fix $p > 2$ and let $w = (w_i)_{i=1}^{\infty}$ be a positive sequence satisfying $w_i \to 0$ and $\sum_{i=1}^{\infty} w_i^{2p/(p-2)} = \infty$. Let $X_{p,w}$ be the space of all sequences $(a_i)_{i=1}^{\infty}$ with

$$\|(a_i)_{i=1}^{\infty}\|_{X_{p,w}} = \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} + \left(\sum_{i=1}^{\infty} |a_i|^2 w_i^2 \right)^{1/2}.$$

It is known that $X_{p,w}$ is isomorphic to a complemented subspace $Y_{p,w}$ of $L_p(I)$ ([8]). Moreover $X_{p,w}$ is isomorphic to the conjugate of X_q $(p^{-1} + q^{-1} = 1)$ which appeared in the proof of the theorem.

It is quite simple to prove that the natural basis of $Y_{p,w}^{\otimes 2}$ is equivalent to the natural basis of the space of all sequences $(a_{i,j})_{i,j=1}^{\infty}$ with:

$$\|(a_{i,j})\| = \left[\sum_{i,j=1}^{\infty} |a_{i,j}|^p + \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{i,j}|^2 w_j^2 \right)^{p/2} + \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |a_{i,j}|^2 w_i^2 \right)^{p/2} + \left(\sum_{i,j=1}^{\infty} |a_{i,j}|^2 w_i^2 w_j^2 \right)^{p/2} \right]^{1/p}.$$

This fact has a natural generalization to $Y_p^{\otimes n}$. We do not write it explicitly to avoid clumsiness of notations.

REMARK 3. Let $p > 2$. All the previously known separable $\mathscr{L}_p$ spaces except $L_p(I)$ itself are isomorphic to a subspace of $(l_2 \oplus l_2 \oplus \cdots)_p$. This latter space has the property that it is isomorphic to the square (in the sense of $\otimes$) of itself. Thus by the method we introduced one cannot "come out" of $(l_2 \oplus l_2 \oplus \cdots)_p$. This raises the following:

PROBLEM 1. Does there exist a separable $\mathscr{L}_p$ space $(p > 2)$ which is neither isomorphic to $L_p(I)$ nor to a subspace of $(l_2 \oplus l_2 \oplus \cdots)_p$?

From our construction it follows only that $X_p^{\otimes n}$ is an $\mathscr{L}_{p,f(n)}$ space where $f(n) \xrightarrow[n\to\infty]{} \infty$. This raises:

PROBLEM 2. Does there exist a $K > 1$ such that there exist infinitely many isomorphic types of separable, infinite dimensional $\mathscr{L}_{p,K}$ spaces?

This problem is obviously connected with:

PROBLEM 3. Let $1 \leqq p < \infty$, $p \neq 2$. Are there uncountably many mutually nonisomorphic, separable $\mathscr{L}_p$ spaces?

REFERENCES

1. W. B. Johnson and E. Odell, *Subspaces of L_p which embed into l_p*, Compositio Math. **28** (1974), 37–49.

2. J. Lindenstrauss, *A remark on $\mathscr{L}_1$ spaces*, Israel J. Math. **8** (1970), 80–82.

3. J. Lindenstrauss and Pelczynski, *Absolutely summing operators in $\mathscr{L}_p$ spaces and their applications*, Studia Máth. **29** (1968), 275–326.

4. J. Lindenstrauss and A. Pelczynski, *Contributions to the theory of the classical Banach spaces*, J. Functional Analysis **8** (1971), 225–249.

5. J. Lindenstrauss and H. P. Rosenthal, *The $\mathscr{L}_p$ spaces*, Israel J. Math. **7** (1969), 325–349.

6. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Springer, 1973.

7. B. Maurey and G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géimétriques des espaces de Banach* (to appear).

8. H. P. Rosenthal, *On the subspaces of L^p ($p > 2$) spanned by sequences of independent random variables*, Israel J. Math. **8** (1970), 273–303.

9. H. P. Rosenthal, *On the span in L^p of sequences of independent random variables* (II), Proc. of the 6th Berkeley Symp. on Prob. and Stat., Berkeley, Calif., 1971, Vol. II, pp. 149–167.

10. J. Y. T. Woo, *On a class of universal modular sequence spaces*, Israel J. Math. **20** (1975), 193–215.

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